EXPECTED NUMBER OF LOCAL MAXIMA OF SOME GAUSSIAN RANDOM POLYNOMIALS

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Abstract

Let $Q_n(x) = \sum_{i=0}^{n} A_i x^i$ be a random algebraic polynomial where the coefficients $A_0, A_1, \cdots$ form a sequence of centered Gaussian random variables. Moreover, assume that the increments $\Delta_j = A_j - A_{j-1}$, $j = 0, 1, 2, \cdots$ are independent, $A_{-1} = 0$. The coefficients can be considered as n consecutive observations of a Brownian motion. We study the asymptotic behaviour of the expected number of local maxima of $Q_n(x)$ below level $u = O(n^k)$, for some $k > 0$.

Keywords and Phrases: random algebraic polynomial, number of real zeros,Local Maxima, expected density, Brownian motion.
AMS(2000) subject classifications. Primary 60H42, Secondary 60G99.

1 Introduction

The theory of the expected number of real zeros of random algebraic polynomials was addressed in the fundamental work of M. Kac[6] (1943). The works Wilkins [12], and Farahmand [3], [5] and Sambandham [10, 11] are other fundamental contributions to the subject. For various aspects on random polynomials see Bharucha-Reid and Sambandham [1], and Farahmand[4].

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There has been recent interest in cases where the coefficients form certain random processes, Rezakhah and Shemehsavar [7], Rezakhah and Soltani [?, 8].

Let $A_0, A_1, \cdots$ be a mean zero Gaussian random sequence for which the increments $\Delta_i = A_i - A_{i-1},\ i = 1, 2, \cdots$ are independent, $A_{-1} = 0$. The sequence $A_0, A_1 \cdots$ may be considered as successive Brownian points, i.e., $A_j = W(t_j),\ j = 0, 1, \cdots$, where $t_0 < t_1 < \cdots$ and $\{W(t),\ t \geq 0\}$ is the standard Brownian motion. In this physical interpretation, $\mathrm{Var}(\Delta_j)$ is the distance between successive times $t_{j-1},\ t_j$. Let

$$Q_n(x) = \sum_{i=0}^{n} A_i x^i,\ \ -\infty < x < \infty, \tag{1.1}$$

We note that $A_j = \Delta_0 + \Delta_1 + \cdots + \Delta_j,\quad j = 0, 1, \cdots$, where $\Delta_i \sim N(0, \sigma_i^2)$ and Δ_i are independent, $i = 0, 1, \cdots$. Thus $Q_n(x) = \sum_{k=0}^{n}(\sum_{j=k}^{n} x^j)\Delta_k = \sum_{k=0}^{n} a_k(x)\Delta_k,$, $Q_n'(x) = \sum_{k=0}^{n} b_k(x)\Delta_k,$, and $Q_n''(x) = \sum_{k=0}^{n} d_k(x)\Delta_k$, where

$$a_k(x) = \sum_{j=k}^{n} x^j,\quad b_k(x) = \sum_{j=k}^{n} jx^{j-1},\ \ d_k(x) = \sum_{j=k}^{n} j(j-1)x^{j-2}\quad k = 0, \cdots, n. \tag{1.2}$$

In this paper we study the asymptotic behavior of the expected number of local maximas of $Q_n(x)$. We say $Q_n(x)$ has a local maxima at $t = t_i$ if $Q_n'(x)$ has a down-crossing of the level zero at t_i. A local maxima which we consider here, is a maxima that occurs when $Q_n(x)$ is below level u. The total number of down-crossing of the level zero by $Q_n'(x)$ in (a, b) is defined as $M(a, b)$, and these occur at the points $a < t_1 < t_2 < \cdots < t_{M(a,b)} < b$. We define $M_u(a, b)$ as the number of zero-down crossing by $Q_n'(x)$ at those points $t_i \in (a,\ b)$, where $Q(t_i) \leq u$.

Rice [1945, pp 71] showed that for any function of the random variables $A_0, A_, \cdots A_n$ and x, say here $U = Q_n(x)$, the expected number of maxima of U in the interval (a, b) is equal to

$$\int_a^b \int_{-\infty}^{\infty} \int_{-\infty}^{0} |t| p_x(r, 0, t)\, dt\, dr\, dx \tag{1.3}$$

where $p_x(r, s, t)$ is the joint probability density function of $U = Q_n(x)$, $V = \partial Q_n(x)/\partial x$, and $W = \partial^2 Q_n(x)/\partial x^2$. Using this formula we find that the expected number of local maxima of $Q_n(x)$ below level u, and inside any interval $(a,\ b), EM_u(a, b)$ is equal to

$$EM_u(a, b) = \int_a^b \int_{-\infty}^{u} \int_{-\infty}^{0} |t| p_x(r, 0, t)\, dt\, dr\, dx \tag{1.4}$$

where

$$p_x(r,0,t) = \frac{\exp(-Lr^2 - 2Mrt - Kt^2)}{(2\pi)^{3/2}\det(\Sigma)^{1/2}}$$

in which Σ is the covariance matrix of (U, V, W), and

$$K = \frac{A^2B^2 - C^2}{2\det(\Sigma)}, \qquad L = \frac{B^2D^2 - F^2}{2\det(\Sigma)} \tag{1.5}$$

$$M = \frac{CF - B^2E}{2\det(\Sigma)}, \qquad S = K - \frac{M^2}{4L}$$

and

$$\det(\Sigma) = A^2B^2D^2 - A^2F^2 - B^2E^2 - C^2D^2 + 2CEF$$

$$A^2 = \mathrm{Var}(Q_n(x)) = \sum_{k=1}^{n} a_k^2(x)\sigma_k^2, \qquad B^2 = \mathrm{Var}(Q_n'(x)) = \sum_{k=1}^{n} b_k^2(x)\sigma_k^2,$$

$$D^2 = \mathrm{Var}(Q_n''(x)) = \sum_{k=1}^{n} d_k^2(x)\sigma_k^2, \qquad C = \mathrm{Cov}(Q_n(x), Q_n'(x)) = \sum_{k=1}^{n} a_k(x)b_k(x)\sigma_k^2,$$

$$E = \mathrm{Cov}(Q_n(x), Q_n''(x)) = \sum_{k=1}^{n} a_k(x)d_k(x)\sigma_k^2,$$

$$F = \mathrm{Cov}(Q_n'(x), Q_n''(x)) = \sum_{k=1}^{n} b_k(x)d_k(x)\sigma_k^2,$$

where $a_k(x)$, $b_k(x)$ and $d_k(x)$ are defined in (2.1).

Using (1.5),and the function $\mathrm{erf}(t) = 2\Phi(t\sqrt{2}) - 1$, , we find that

$$EM_u(a,b) = \int_a^b f_n(x)dx = J_1 + J_2 \tag{1.6}$$

$$J_1 = \frac{1}{4\pi}\int_a^b G_1[\mathrm{erf}(G_2) + 1]dx, \qquad J_2 = -\frac{1}{4\pi}\int_a^b G_1G_3\Big[\mathrm{erf}(G_4) + 1\Big]\exp(G_5)$$

where $G_1 = \left(2S\sqrt{2L\det(\Sigma)}\right)^{-1}$, $G_2 = u\sqrt{L}$, $G_4 = u\sqrt{K^{-1}M^2}$,

$$G_3 = \sqrt{\frac{M^2}{LK}}, \qquad G_5 = -\frac{LSu^2}{K}$$

Farahmand[5] obtained a similar formula for the case where the coefficients are independent, normally distributed with mean zero and variance one.

2 Asymptotic behaviour of EM_u

In this section we obtain the asymptotic behaviour of the expected number of local maxima of $Q_n(x) = 0$ given by (1.1). We prove the following theorem for the case that the increments $\Delta_1 \cdots \Delta_n$ are independent and have the same distribution. Also we assume that $\sigma_k^2 = 1$, for $k = 1 \cdots n$.

Theoream(2,1): Let $Q_n(x)$ be the random algebraic polynomial given by (1.1) for which $A_j = \Delta_1 + ... + \Delta_j$ where $\Delta_i, i = 1, ..., n$ are independent and $\Delta_j \sim N(0,1)$ then the expected number of local maxima of $Q_n(x)$ below level u satisfies:

$$EM_u(1,\infty) = \frac{0.0013074}{4\pi} + \frac{(0.0350655)u}{2(n\pi)^{3/2}} + O(n^{-1/2}) \qquad \text{for } u = O(n^{5/4})$$

$$EM_u(0,1) = \frac{2(\sqrt{35}-5)}{345\pi}\ln\Big(\frac{n^{3/2}}{u}\Big) - 0.001648 - \frac{(2.033388)u}{2(n\pi)^{3/2}} + O(n^{-1/2}) \quad \text{for } u = O(n^{5/4})$$

$$EM_u(-\infty,-1) = \frac{0.0162552}{4\pi} + \frac{(0.0997677)u}{2\pi\sqrt{n\pi}} + O(n^{-1/2}) \qquad \text{for } u = O(n^{1/4})$$

$$EM_u(-1,0) = \frac{2(\sqrt{3}-1)}{11\pi}\ln\Big(\frac{n^{1/2}}{u}\Big) + 0.081413 - \frac{(0.594923)u}{2\pi\sqrt{n\pi}} + O(n^{-1/2}) \quad \text{for } u = O(n^{1/4})$$

proof: The asymptotic behaviour is treated separately on the intervals $1 < x < \infty$, $-\infty < x < -1$, $0 < x < 1$ and $-1 < x < 0$.

For $u = O(n^{5/4})$ and $1 < x < \infty$, by the change of variable $x = 1 + \frac{t}{n}$ and the equality $\left(1 + \frac{t}{n}\right)^n = e^t\left(1 - \frac{t^2}{n}\right) + O\left(\frac{1}{n^2}\right)$. Using (1.6), we find that

$$EM_u(1,\infty) = \frac{1}{n}\int_0^\infty f_n(1 + \frac{t}{n})dt,$$

where by (1.5) and (1.6), and by tedious manipulation we have that

$$n^{-1}G_1\left(1 + \frac{t}{n}\right) = H_{11}(t) + O(n^{-1}), \;\; G_3\left(1 + \frac{t}{n}\right) = H_{13}(t) + O(n^{-1}) \tag{2.1}$$

$$G_2\left(1 + \frac{t}{n}\right) = \frac{2u}{n^{3/2}\sqrt{\pi}}H_{12}(t) + O(n^{-5/4}),$$

$$G_4\left(1 + \frac{t}{n}\right) = \frac{2u}{n^{3/2}\sqrt{\pi}}H_{14}(t) + O(n^{-5/4}), \;\; G_5\left(1 + \frac{t}{n}\right) = 1 + O(n^{-1/2}),$$

where

$$H_{11}(t) = \frac{1}{192}\Big(-4 + \left(32t + 16 + 32t^2\right)e^t$$

$$
+\left(32t^5+208t^4+472t+124+1040t^2+736t^3\right)e^{2t}
$$

$$
+\left(-288+192t^4-768t+256t^3-1152t^2\right)e^{3t}
$$

$$
+\left(20-80t^2+16t+176t^4-64t^5-704t^3\right)e^{4t}
$$

$$
+\left(272+224\,t+160t^2\right)e^{5t}+\left(-140+24t\right)e^{6t}\Big)\times
$$

$$
\Bigg(35-\left(32-32t+160t^2\right)e^t-(294-588t-324t^2+600t^3+216t^4+80t^5)e^{2t}
$$

$$
+(544-1632t+1056t^2+512t^3-192t^4)e^{3t}
$$

$$
+\left(1012t-253-1400t^2+736t^3-172t^4+16t^5\right)e^{4t}\Bigg)^{1/2}\times
$$

$$
\Bigg[\frac{115}{192}+\left(-\frac{7}{8}-\frac{35}{12}t^2+\frac{37}{8}t\right)e^t
$$

$$
-\left(\frac{733}{48}+\frac{485}{24}t+\frac{523}{48}t^2+\frac{1073}{24}t^3+\frac{5}{3}t^5+\frac{73}{12}t^4\right)e^{2t}
$$

$$
+\left(\frac{1043}{24}-\frac{232}{3}t^4-\frac{125}{8}t+\frac{364}{3}t^3-\frac{34}{3}t^6+162t^2-14t^5\right)e^{3t}
$$

$$
+\left(43t^7-\frac{32777}{48}t^2+\frac{2161}{12}t^5+\frac{31}{3}t^8+\frac{507}{8}t+\frac{5177}{16}t^4+\frac{1239}{32}+\frac{1949}{12}t^6+\frac{1043}{12}t^3\right)e^{4t}
$$

$$
+\left(-\frac{6887}{24}+\frac{7153}{24}t+\frac{12731}{12}t^2-\frac{5627}{6}t^3+26t^7-\frac{2335}{6}t^4-\frac{173}{3}t^6-\frac{1697}{6}t^5\right)e^{5t}
$$

$$
+\left(\frac{20383}{48}+\frac{68}{3}t^7-\frac{4023}{16}t^2-\frac{8}{3}t^8-\frac{7297}{8}t+\frac{29851}{24}t^3-\frac{3907}{24}t^4-\frac{547}{6}t^6+\frac{397}{4}t^5\right)e^{6t}
$$

$$
+\left(-\frac{2018}{3}t^2-\frac{2141}{8}+\frac{527}{2}t^4+6t^6+\frac{6749}{8}t-\frac{1243}{6}t^3-\frac{401}{6}t^5\right)e^{7t}
$$

$$
+\left(\frac{12155}{192}-\frac{6281}{24}t+\frac{1385}{16}t^4+\frac{19097}{48}t^2+t^6-\frac{787}{3}t^3-\frac{29}{2}t^5\right)e^{8t}\Bigg]^{-1}t^{-1}
$$

and

$$
H_{12}(t)=\Bigg[-80\,\Bigg((\frac{253}{80}-\frac{46}{5}t^3+\frac{35}{2}t^2-1/5t^5+\frac{43}{20}t^4-\frac{253}{20}t)e^{-2t}
$$

$$
+(\frac{147}{40}+\frac{27}{10}t^4+t^5-\frac{81}{20}t^2+15/2t^3-\frac{147}{20}t)e^{-4t}
$$

$$
+(\frac{102}{5}t-\frac{34}{5}-\frac{66}{5}t^2-\frac{32}{5}t^3+\frac{12}{5}t^4)e^{-3t}-\frac{7}{16}e^{-6t}
$$

$$
+(2t^2-2/5t+2/5)e^{-5t}\Bigg)\,t^3
$$

$$\times\Big(\Big(-176\,t^3-20t^2-16t^5+44t^4+4t+5\Big)\,e^{-2t}$$
$$+\Big(184t^3+260t^2+52t^4+31+118t+8t^5\Big)\,e^{-4t}$$
$$+\Big(64t^3-192\,t-288t^2-72+48t^4\Big)\,e^{-3t}$$
$$+\Big(56t+68+40t^2\Big)\,e^{-t}+6t-35-e^{-6t}+\Big(8\,t+8t^2+4\Big)\,e^{-5t}\Big)^{-1}\Big]^{1/2}$$

Also

$$H_{13}(t)=\Big(5+\Big(4+36t-8t^2\Big)\,e^{t}-\Big(12t-24t^4-52t^3+78-150t^2\Big)\,e^{2t}$$
$$-\Big(120t^2-124-24t^3+156t\Big)\,e^{3t}+\Big(132t-58t^2+8t^3-55\Big)\,e^{4t}\Big)$$
$$\times\Big[\Big(\Big(253-736t^3+172t^4+1400t^2-1012t-16t^5\Big)\,e^{4t}$$
$$+\Big(294-324t^2-588t+600t^3+80t^5+216t^4\Big)\,e^{2\,t}$$
$$-\Big(512t^3+1056t^2+544-1632t-192t^4\Big)\,e^{3t}-35+\Big(160t^2+32-32t\Big)\,e^{t}\Big)$$
$$\times\Big((15-4t)e^{4t}-(32+24t)e^{3t}+\Big(36t+18+12t^2+8t^3\Big)\,e^{2t}-8e^{t}t-1\Big)\Big]^{-1/2}$$

and

$$H_{14}(t)=\Big(\Big(-78+150t^2+52t^3-12t+24\,t^4\Big)\,e^{-3t}$$
$$+\Big(-120t^2-156t+124+24t^3\Big)\,e^{-2t}$$
$$+\Big(132t+8\,t^3-58t^2-55\Big)\,e^{-t}+5\,e^{-5t}+\Big(36\,t+4-8t^2\Big)\,e^{-4t}\Big)\,t^{3/2}$$
$$\times\Big((4t-15)+(32+24t)e^{-t}-(18+36t+12t^2+8t^3)e^{-2t}+8te^{-3t}+e^{-4t}\Big)^{-1/2}$$
$$\Big(-35+6t+(68+56t+40t^2)e^{-t}+(5+4t-20t^2-176t^3+44t^4-16t^5)e^{-2t}$$
$$+(64t^3-72-192t+48t^4-288t^2)e^{-3t}+(31+118t+260t^2+184t^3+8t^5+52t^4)e^{-4t}$$
$$+(4+8t+8t^2)e^{-5t}-e^{-6t}\Big)^{-1/2}$$

As $t\to\infty$ we have that

$$H_{11}(t)\sim\frac{1}{2t^{7/2}},\qquad H_{13}(t)\sim 1,\qquad H_{12}(t)=O(t^{7/2}e^{-t}),\qquad H_{14}(t)=O(t^{7/2}e^{-t})$$

Thus by (2.1) and above calculations we have that

$$
\begin{aligned}
EM_u(1,\infty) &= \frac{1}{n}\int_0^\infty f_n\Big(1+\frac{t}{n}\Big)dt \\
&= \frac{1}{4\pi}\int_0^\infty H_{11}(t)dt + \frac{u}{2n^{3/2}\pi\sqrt{\pi}}\int_0^\infty H_{11}(t)H_{12}(t)dt \\
&\quad -\frac{u}{2n^{3/2}\pi\sqrt{\pi}}\int_0^\infty H_{11}(t)H_{13}(t)H_{14}(t)dt - \frac{1}{4\pi}\int_0^\infty H_{11}(t)H_{13}(t)dt\Big]
\end{aligned}
$$

where$\int_0^\infty H_{11}(t)dt = 0.02789960660$ and $\int_0^\infty H_{11}(t)H_{13}(t)dt = 0.02659218098$ Also $\int_0^\infty H_{11}(t)H_{12}(t)dt = 0.3326450540$ and $\int_0^\infty H_{11}(t)H_{13}(t)H_{14}(t)dt = 0.297579554$

For $u = O(n^{1/4})$ and $-\infty < x < -1$, let $x = -1-\frac{t}{n}$ then, by (1.6), $EM_u(-\infty,-1) = \frac{1}{n}\int_0^\infty f_n\left(-1-\frac{t}{n}\right)dt$. Using (1.5), (1.6) we have that

$$
n^{-1}G_1\left(-1-\frac{t}{n}\right) = H_{21}(t)+O(n^{-1}), \qquad G_2\left(-1-\frac{t}{n}\right) = \frac{2u}{\sqrt{n\pi}}H_{22}(t)+O(n^{-5/4}), \quad (2.2)
$$

$G_3\left(-1-\frac{t}{n}\right) = H_{23}(t)+O(n^{-1})$, and

$$
G_4\left(-1-\frac{t}{n}\right) = \frac{2u}{\sqrt{n\pi}}H_{24}(t)+O(n^{-5/4}), \qquad G_5\left(-1-\frac{t}{n}\right) = 1+O(n^{-1/2}),
$$

where

$$
\begin{aligned}
H_{21}(t) &= \frac{3}{2t}\Big(\big(-3/8-2\,t^5-9/2\,t^2-2\,t^3+3/2\,t^4-3/2\,t\big)e^{4\,t} \\
&+\big(t^5+9/2\,t^4+3/4\,t+7\,t^3+9/2\,t^2+3/8\big)e^{2\,t}+3/4\,e^{6\,t}t+1/8\,e^{6\,t}-1/8\Big) \\
&\Big[3+(12t-6-12t^2-56t^3-56t^4-16t^5)e^{2t}+(3-12t+24t^2+32t^3-44t^4+16t^5)e^{4t}\Big]^{1/2} \\
&\Big[\Big(7/3t^8+\frac{221}{12}t^5+1/8t+\frac{11}{32}+\frac{259}{48}t^4+7/4t^3+\frac{35}{3}t^7+\frac{257}{12}t^6+\frac{29}{16}t^2\Big)e^{4\,t} \\
&+\Big(-\frac{15}{8}t^3-\frac{55}{48}t^2-1/24\,t-5/4t^4-1/3t^5-\frac{11}{48}\Big)e^{2t} \\
&+\Big(\frac{13}{6}t^6-1/8t-\frac{49}{24}t^3-\frac{51}{4}t^5-8/3t^8+8/3t^7-\frac{211}{24}t^4-\frac{11}{48}-3/16t^2\Big)e^{6t} \\
&+\frac{11}{192}+\Big(-5/2t^5+1/24t-\frac{23}{48}t^2+t^6+\frac{13}{6}t^3+\frac{11}{192}+\frac{25}{16}t^4\Big)e^{8t}\Big]^{-1},
\end{aligned}
$$

$$
\begin{aligned}
H_{22}(t) &= 2\sqrt{t}\Big(3-(16t^5+56t^4+56t^3+12t^2-12t+6)e^{2t} \\
&\quad +(16t^5-44t^4+32t^3+24t^2-12t+3)e^{4t}\Big)^{1/2}
\end{aligned}
$$

$$\times\Big[(1+6t)e^{6t}+(12t^4-16t^5-16t^3-36t^2-12t-3)e^{4t}+(3+6t+36t^2+56t^3+36t^4+8t^5)e^{2t}-1\Big]^{-1/2}$$

and

$$\begin{aligned}
H_{23}(t) = \frac{1}{8}\Big[\Big(1 + (8t^4 + 20t^3 + 14t^2 + 4t - 2)e^{2t} + (1 - 4t - 10t^2 + 8t^3)e^{4t}\Big)^2 \\
\times\Big(\big((-2t^3 - 3t^2 - t - 1/2)e^{2t} + 1/4 + (1/4 + t)e^{4t}\big) \\
\times \big((t^5 - 11/4t^4 + 3/16 + 2t^3 + 3/2t^2 - 3/4t)e^{4t} + 3/16 \\
+(3/4t - 3/4t^2 - 7/2t^3 - 7/2t^4 - t^5 - 3/8)e^{2t}\big) \Big)^{-1}\Big]^{1/2},
\end{aligned}$$

$$\begin{aligned}
H_{24}(t) = 2\sqrt{t}\Big(1 + (8t^4 + 20t^3 + 14t^2 + 4t - 2)e^{2t} + (1 - 4t - 10t^2 + 8t^3)e^{4t}\Big) \\
\times\Big[\Big((3 + 6t + 36t^2 + 56t^3 + 36t^4 + 8t^5)e^{2t} + (1 + 6t)e^{6t} - 1 \\
+(12t^4-16t^5-16t^3-36t^2-12t-3)e^{4t}\Big)\Big(1 - (2 + 4t + 12t^2 + 8t^3)e^{2t} + (1 + 4t)e^{4t}\Big)\Big]^{-1/2}
\end{aligned}$$

As $t \to \infty$ we have that

$$H_{21}(t) \sim \frac{1}{2t^{7/2}}, \qquad H_{23}(t) \sim 1, \qquad H_{22}(t) = O(t^{5/2}e^{-t}), \qquad H_{24}(t) = O(t^{5/2}e^{-t})$$

Thus by (2.2) and above calculations we have that

$$\begin{aligned}
EM_u(-\infty, -1) &= \frac{1}{n}\int_0^\infty f_n(-1 - \frac{t}{n}) \\
&= \frac{1}{4\pi}\int_0^\infty H_{21}(t)dt + \frac{u}{2\pi\sqrt{n\pi}}\int_0^\infty H_{21}(t)H_{22}(t)dt \\
&\quad -\frac{u}{2\pi\sqrt{n\pi}}\int_0^\infty H_{21}(t)H_{23}(t)H_{24}(t)dt - \frac{1}{4\pi}\int_0^\infty H_{21}(t)H_{23}(t)dt
\end{aligned}$$

where $\int_0^\infty H_{21}(t)dt = .10652624145$ and $\int_0^\infty H_{21}(t)H_{23}(t)dt = .090270992310$. Also $\int_0^\infty H_{21}(t)H_{22}(t)dt = 0.3240703564$ and $\int_0^\infty H_{21}(t)H_{23}(t)H_{24}(t)dt = 0.2243026030$

For $u = O(n^{5/4})$ and $0 < x < 1$, let $x = 1 - \frac{t}{n+t}$. Thus by (1.6), $EM_u(0,1) = \left(\frac{n}{(n+t)^2}\right)\int_0^\infty f_n\left(1 - \frac{t}{n+t}\right)dt$, where by (1.5) and (1.6) we have that

$$\frac{n}{(n+t)^2}G_1\Big(1-\frac{t}{n+t}\Big)=H_{31}(t)+O(n^{-1}), \quad G_2\left(1-\frac{t}{n+t}\right)=\frac{2u}{n^{3/2}\sqrt{\pi}}H_{32}(t)+O(n^{-5/4}),$$

$$G_3\left(1 - \frac{t}{n+t}\right) = H_{33}(t) + O(n^{-1}), \qquad G_5\left(1 - \frac{t}{n+t}\right) = 1 + O(n^{-1/2}), \tag{2.3}$$

$$G_4\left(1-\frac{t}{n+t}\right)=\frac{2u}{n^{3/2}\sqrt{\pi}}H_{34}(t)+O(n^{-5/4});$$

where

$$H_{31}(t)=\frac{-1}{31t}\Big((-1/4t-5/4t^2+11t^3+t^5+\frac{5}{16}+11/4t^4)e^{-4t}$$
$$+(-23/2t^3-1/2t^5+\frac{13}{4}t^4+\frac{65}{4}t^2-\frac{59}{8}t+\frac{31}{16})e^{-2t}$$
$$+(12t-4t^3-18t^2+3t^4-9/2)e^{-3t}+(-\frac{35}{16}-3/8t)e^{-6t}+$$
$$(5/2t^2-7/2t+\frac{17}{4})e^{-5t}-1/16+(1/2t^2-1/2t+1/4)e^{-t}\Big)$$
$$\times\Big(35-(32+32t+160t^2)e^{-t}+(80t^5-216t^4+600t^3+324t^2-588t-294)e^{-2t}$$
$$+(544+1632t+1056t^2-512t^3-192t^4)e^{-3t}-(253+1012t+1400t^2+736t^3+172t^4+16t^5)e^{-4t}\Big)^{1/2}$$
$$\times\Big((-\frac{2161}{124}t^5-\frac{129}{31}t^7+\frac{1949}{124}t^6-\frac{1043}{124}t^3+t^8+\frac{3717}{992}-\frac{1521}{248}t-\frac{32777}{496}t^2+\frac{501}{16}t^4)e^{-4t}$$
$$+(\frac{1073}{248}t^3+\frac{5}{31}t^5-\frac{73}{124}t^4-\frac{523}{496}t^2+\frac{485}{248}t-\frac{733}{496})e^{-2t}$$
$$+(-\frac{232}{31}t^4+\frac{375}{248}t-\frac{364}{31}t^3+\frac{486}{31}t^2+\frac{1043}{248}-\frac{34}{31}t^6+\frac{42}{31}t^5)e^{-3t}$$
$$+(\frac{21891}{248}t+\frac{20383}{496}-\frac{8}{31}t^8-\frac{68}{31}t^7-\frac{3907}{248}t^4-\frac{547}{62}t^6-\frac{29851}{248}t^3-\frac{12069}{496}t^2-\frac{1191}{124}t^5)e^{-6t}$$
$$+(\frac{12731}{124}t^2-\frac{7153}{248}t-\frac{6887}{248}+\frac{1697}{62}t^5-\frac{2335}{62}t^4+\frac{5627}{62}t^3-\frac{173}{31}t^6-\frac{78}{31}t^7)e^{-5t}$$
$$+(-\frac{35}{124}t^2-\frac{111}{248}t-\frac{21}{248})e^{-t}$$
$$+(\frac{4155}{496}t^4+\frac{6281}{248}t+\frac{3}{31}t^6+\frac{787}{31}t^3+\frac{12155}{1984}+\frac{19097}{496}t^2+\frac{87}{62}t^5)e^{-8t}$$
$$+\frac{115}{1984}+\left(\frac{1243}{62}t^3+\frac{18}{31}t^6-\frac{20247}{248}t-\frac{2018}{31}t^2+\frac{401}{62}t^5-\frac{6423}{248}+\frac{51}{2}t^4\right)e^{-7t}\Big)^{-1},$$

$$H_{32}(t)=\sqrt{160}t^{3/2}\Big((\frac{253}{40}t+\frac{23}{5}t^3+\frac{35}{4}t^2+\frac{253}{160}+\frac{43}{40}t^4+1/10t^5)e^{-4t}$$
$$+(\frac{147}{80}+\frac{27}{20}t^4-\frac{81}{40}t^2-1/2t^5+\frac{147}{40}t-\frac{15}{4}t^3)e^{-2t}$$
$$+(\frac{16}{5}t^3+6/5t^4-\frac{51}{5}t-\frac{33}{5}t^2-\frac{17}{5})e^{-3t}-\frac{7}{32}+(1/5+1/5t+t^2)e^{-t}\Big)^{1/2}$$
$$\times\Big((16t^5-4t+176t^3-20t^2+44t^4+5)e^{-4t}$$

$$+(260t^2+31+52t^4-184t^3-118t-8t^5)e^{-2t}$$

$$+(-72-288t^2+192t+48t^4-64t^3)e^{-3t}+(-8t+4+8t^2)e^{-t}$$

$$+(68-56t+40t^2t)e^{-5t}-1-6e^{-6t}t-35e^{-6t}\Big)^{-1/2}$$

and

$$H_{33}(t)=1/20\sqrt{20}\Bigg(\Bigg(\Big(13/2t^3-3/2t-\frac{75}{4}t^2+\frac{39}{4}-3t^4\Big)e^{-2t}$$

$$+\Big(t^3+\frac{29}{4}t^2+\frac{55}{8}+\frac{33}{2}t\Big)e^{-4t}+\Big(-\frac{39}{2}t+15t^2-\frac{31}{2}+3t^3\Big)e^{-3t}-5/8$$

$$+\Big(-1/2+9/2t+t^2\Big)e^{-t}\Bigg)^2\Bigg(\Big(9/4-9/2t+3/2t^2-t^3\Big)e^{-2t}$$

$$+(3t-4)e^{-3t}+(15/8+1/2t)e^{-4t}+te^{-t}-1/8\Bigg)^{-1}$$

$$\times\Bigg(\Big(\frac{27}{20}t^4-\frac{15}{4}t^3-\frac{81}{40}t^2+\frac{147}{80}-1/2t^5+\frac{147}{40}t\Big)e^{-2t}$$

$$+\Big(\frac{23}{5}t^3+\frac{253}{160}+\frac{43}{40}t^4+1/10t^5+\frac{253}{40}t+\frac{35}{4}t^2\Big)e^{-4t}$$

$$+\Big(6/5t^4-\frac{17}{5}-\frac{51}{5}t-\frac{33}{5}t^2+\frac{16}{5}t^3\Big)e^{-3t}-\frac{7}{32}+\Big(t^2+1/5t+1/5\Big)e^{-t}\Bigg)^{-1}\Bigg)^{1/2}$$

$$H_{34}(t)=\frac{1}{\sqrt{64}}\Big(5+\Big(4-36t-8t^2\Big)e^{-t}+\Big(24t^4-52t^3+150t^2+12t-78\Big)e^{-2t}$$

$$+\Big(156t-24t^3-120t^2+124\Big)e^{-3t}-\Big(8t^3+58t^2+132t+55\Big)e^{-4t}\Big)t^{3/2}$$

$$\times\Bigg(\Bigg(\Big(-\frac{65}{2}t^2+\frac{59}{4}t+23t^3-\frac{31}{8}+t^5-13/2t^4\Big)e^{-2t}$$

$$+\Big(1/2t-22t^3+5/2t^2-11/2t^4-2t^5-5/8\Big)e^{-4t}+\Big(-24t+8t^3+9+36t^2-6t^4\Big)e^{-3t}$$

$$+\Big(-t^2-1/2+t\Big)e^{-t}+\Big(-5t^2-17/2+7t\Big)e^{-5t}+1/8+(35/8+3/4t)e^{-6t}\Bigg)$$

$$\times\Bigg(\Big(t^3+9/2\,t-9/4-3/2\,t^2\Big)e^{-2t}-te^{-t}+(4-3t)e^{-3t}-(15/8+1/2t)e^{-4t}+1/8\Bigg)\Bigg)^{-1/2}$$

As $t\to\infty$ we have that

$$H_{31}(t)\sim\frac{4\sqrt{35}}{115t},\qquad H_{33}(t)\sim\frac{5}{\sqrt{35}},\qquad H_{32}(t)\sim\sqrt{35}\,t^{3/2},\qquad H_{34}(t)\sim 5t^{3/2}.$$

For any real numbers A and B we have that

$$\frac{A}{t} - \frac{B\sqrt{t}}{n^{3/2}} = \frac{A}{t} - \frac{B\sqrt{t}}{n^{3/2} + (B/A)t^{3/2}} + O(n^{-3}). \tag{2.4}$$

Let $a = \frac{\sqrt{35}}{115\pi}$ and $b = \frac{10u}{23\pi^{3/2}}$, $c = \frac{1}{23\pi}$ and $d = \frac{14u}{23\pi^{3/2}}$. Now by (2.3), (2.4), and above calculations we have that

$$\begin{aligned}
EM_u(0,1) &= \frac{n}{(n+t)^2}\int_0^\infty f_n(1-\frac{t}{n+t})dt = \frac{1}{4\pi}\int_0^\infty \left(H_{31}(t) - \frac{4\sqrt{35}I_{[t\geq 1]}}{115t}\right)dt \\
&+\frac{u}{2(n\pi)^{3/2}}\int_0^\infty \left(H_{31}(t)H_{32}(t) - \frac{28\sqrt{t}I_{[t\geq 1]}}{23}\right)dt \\
&-\frac{1}{4\pi}\int_0^\infty \left(H_{31}(t)H_{33}(t) - \frac{4I_{[t\geq 1]}}{23t}\right)dt \\
&-\frac{u}{2(n\pi)^{3/2}}\int_0^\infty \left(H_{31}(t)H_{33}(t)H_{34}(t) - \frac{20\sqrt{t}I_{[t\geq 1]}}{23}\right)dt \\
&+\int_1^\infty \frac{a}{t} - \frac{b\sqrt{t}}{n^{3/2}+(b/a)t^{3/2}}dt - \int_1^\infty \frac{c}{t} - \frac{d\sqrt{t}}{n^{3/2}+(d/c)t^{3/2}}dt + O(n^{-3})
\end{aligned}$$

where

$$\int_0^\infty\left(H_{31}(t) - \frac{4\sqrt{35}I_{[t\geq 1]}}{115t}\right)dt = -0.2545810,\ \int_0^\infty\left(H_{11}(t)H_{33}(t) - \frac{4I_{[t\geq 1]}}{23t}\right)dt = -0.2085374$$

Also

$$\int_0^\infty \left(H_{31}(t)H_{32}(t) - \frac{28\sqrt{t}I_{[t\geq 1]}}{23}\right)dt = -4.808177963,$$

$$\int_0^\infty \left(H_{31}(t)H_{33}(t)H_{34}(t) - \frac{20\sqrt{t}I_{[t\geq 1]}}{23}\right)dt = -2.774789804$$

and by the above assumption for a, b, c, and d we have that

$$\int_1^\infty \frac{a}{t} - \frac{b\sqrt{t}}{n^{3/2}+\frac{b}{a}t^{3/2}}dt = \frac{2a}{3}\ln\left(\frac{a}{b}n^{3/2}+1\right) = \frac{2\sqrt{35}}{345\pi}\ln\left(\frac{\sqrt{35\pi}}{50u}n^{3/2}+1\right),$$

$$\int_1^\infty \left(\frac{c}{t} - \frac{d\sqrt{t}}{n^{3/2}+(d/c)t^{3/2}}\right)dt = \frac{2c}{3}\ln\left(\frac{c}{d}n^{3/2}+1\right) = \frac{2}{69\pi}\ln\left(\frac{\sqrt{\pi}}{14u}n^{3/2}+1\right)$$

For $u = O(n^{1/4})$ and $-1 < x < 0$, let $x = -1+\frac{t}{n+t}$. Thus by (1.6), $EM_u(-1,0) = \left(\frac{n}{(n+t)^2}\right)\int_0^\infty f_n\left(-1+\frac{t}{n+t}\right)dt$, where by (1.6), (1.5) we have that

$$\frac{n}{(n+t)^2}G_1\left(-1+\frac{t}{n+t}\right) = H_{41}(t)+O(n^{-1}),\quad G_2\left(-1+\frac{t}{n+t}\right) = \frac{2u}{\sqrt{n\pi}}H_{42}(t)+O(n^{-5/4}),$$

$$G_3\left(-1+\frac{t}{n+t}\right)=H_{43}(t)+O(n^{-1}),\quad G_5\left(-1+\frac{t}{n+t}\right)=1+O(n^{-1/2}), \tag{2.5}$$

$$G_4\left(-1+\frac{t}{n+t}\right)=\frac{2u}{\sqrt{n\pi}}H_{44}(t)+O(n^{-5/4}),$$

where

$$H_{41}(t)=\frac{1}{16t}\Big((2t^5-\frac{31}{2}t^4-\frac{9}{8}-\frac{9}{2}t+\frac{3}{2}t^2+12t^3)e^{-4t}+(7t^3-\frac{9}{2}t^2+t^5-\frac{9}{2}t^4+\frac{3}{8}+\frac{3}{4}t)e^{-2t}$$
$$+\frac{1}{8}+(6t^2-4t^5-8t^3-11t^4+\frac{15}{4}t+\frac{5}{8})e^{-6t}\Big)$$
$$\times\Big(\Big(3+(12t-32t^3+24t^2-44t^4-16t^5+3)e^{-4t}+(16t^5-56t^4+56t^3-12t^2-12t-6)e^{-2t}\Big)\Big)^{1/2}$$
$$\times\Big((\frac{47}{32}t^3+\frac{15}{128}t^2-\frac{205}{32}t^5+\frac{257}{32}t^6-\frac{39}{256}-\frac{35}{8}t^7+\frac{7}{8}t^8-\frac{39}{64}t+\frac{35}{128}t^4)e^{-4t}$$
$$+(\frac{45}{64}t^3-\frac{55}{128}t^2+1/8t^5-\frac{15}{32}t^4+\frac{1}{64}t+\frac{1}{128})e^{-2t}$$
$$+(\frac{279}{128}t^2+\frac{73}{32}t^5+\frac{25}{128}-\frac{79}{64}t^3-\frac{459}{64}t^4+\frac{75}{64}t+t^8+\frac{165}{16}t^6-9t^7)e^{-6t}$$
$$+\frac{11}{512}+(\frac{507}{128}t^4+1/8t^6+\frac{147}{16}t^5-\frac{239}{128}t^2-\frac{37}{512}-9/2t^7-\frac{37}{64}t-\frac{27}{16}t^3-2t^8)e^{-8t}\Big)^{-1},$$

and

$$H_{42}(t)=2\sqrt{t}\Big((6+12t+12t^2-56t^3+56t^4-16t^5)e^{-2t}$$
$$+(16t^5+44t^4+32t^3-24t^2-12t-3)e^{-4t}-3\Big)^{1/2}$$
$$\times\Big[(36t^4-8t^5-56t^3+36t^2-6t-3)e^{-2t}+(9+36t-12t^2-96t^3+124t^4-16t^5)e^{-4t}$$
$$+(32t^5+88t^4+64t^3-48t^2-30t-5)e^{-6t}-1\Big]^{-1/2},$$

$$H_{43}(t)=8\Big((-1/4-1/2t-5/2t^3+t^4+7/4t^2)e^{-2t}+1/8+(1/2t-5/4t^2-t^3+1/8)e^{-4t}\Big)$$
$$\times\Big[\Big(3+(16t^5-56t^4+56t^3-12t^2-12t-6)e^{-2t}+(3+12t+24t^2-32t^3-44t^4-16t^5)e^{-4t}\Big)$$
$$\times\Big(1+(2+4t-12t^2+8t^3)e^{-2t}+(16t^3-8t^2-12t-3)e^{-4t}\Big)\Big]^{-1/2}$$

$$H_{44}(t)=2\sqrt{t}\left(1+(8t^4-20t^3+14t^2-4t-2)e^{-2t}+(4t-10t^2-8t^3+1)e^{-4t}\right)$$
$$\times\Big[\Big((3+56t^3+8t^5-36t^4+6t-36t^2)e^{-2t}+(16t^5+12t^2-9+96t^3-36t-124t^4)e^{-4t}$$

$$+(30t-88t^4+5+48t^2-32t^5-64t^3)e^{-6t}+1\Big)$$

$$\times\Big((3+12t+8t^2-16t^3)e^{-4t}-(2+4t-12t^2+8t^3)e^{-2t}-1\Big)\Big]^{-1/2}$$

As $t\to\infty$ we have that

$$H_{41}(t)\sim\frac{4\sqrt{3}}{11t},\qquad H_{43}(t)\sim\frac{1}{\sqrt{3}},\qquad H_{42}(t)\sim 2\sqrt{3t},\qquad H_{44}(t)\sim 2\sqrt{t}.$$

For any real numbers A and B we have that

$$\frac{A}{t}-\frac{B/\sqrt{t}}{n^{1/2}}=\frac{A}{t}-\frac{B/\sqrt{t}}{n^{1/2}+(B/A)t^{1/2}}+O(n^{-1}). \tag{2.6}$$

Let $a=\frac{\sqrt{3}}{11\pi}$ and $b=\frac{4u}{11\pi^{3/2}}$, $c=\frac{1}{11\pi}$ and $d=\frac{12u}{11\pi^{3/2}}$. Now by (2.5), (2.6) and above calculations we have that

$$\begin{aligned}EM_u(-1,0)=&\frac{n}{(n+t)^2}\int_0^\infty f_n(-1+\frac{t}{n+t})dt=\frac{1}{4\pi}\int_0^\infty\left(H_{41}(t)-\frac{4\sqrt{3}I_{[t\geq1]}}{11t}\right)dt\\ &+\frac{u}{2\pi\sqrt{n\pi}}\int_0^\infty\left(H_{41}(t)H_{42}(t)-\frac{24I_{[t\geq1]}}{11\sqrt{t}}\right)dt-\frac{1}{4\pi}\int_0^\infty\left(H_{41}(t)H_{43}(t)-\frac{4I_{[t\geq1]}}{11t}\right)dt\\ &-\frac{u}{2\pi\sqrt{n\pi}}\int_0^\infty\left(H_{41}(t)H_{43}(t)H_{44}(t)-\frac{8I_{[t\geq1]}}{11\sqrt{t}}\right)dt\\ &+\int_1^\infty\frac{a}{t}-\frac{b/\sqrt{t}}{n^{1/2}+(b/a)t^{1/2}}dt-\int_1^\infty\frac{c}{t}-\frac{d/\sqrt{t}}{n^{1/2}+(d/c)t^{1/2}}dt+O(n^{-1})\end{aligned}$$

where $\int_0^\infty\left(H_{41}(t)-\frac{4\sqrt{3}I_{[t\geq1]}}{11t}\right)dt=-0.1146419848$, and $\int_0^\infty\left(H_{41}(t)H_{43}(t)-\frac{4I_{[t\geq1]}}{11t}\right)dt=-0.0801100983$. Also
$\int_0^\infty\left(H_{41}(t)H_{42}(t)-\frac{24I_{[t\geq1]}}{11\sqrt{t}}\right)dt=-0.7769335$, $\int_0^\infty\left(H_{41}(t)H_{43}(t)H_{44}(t)-\frac{8I_{[t\geq1]}}{11\sqrt{t}}\right)dt=-0.1820104$,
and by the above assumption for a, b, c, and d we have that

$$\int_1^\infty\frac{a}{t}-\frac{b/\sqrt{t}}{n^{1/2}+\frac{b}{a}t^{1/2}}dt=2a\ln\Big(\frac{a}{b}n^{1/2}+1\Big)=\frac{2\sqrt{3}}{11\pi}\ln\Big(\frac{\sqrt{3\pi}}{4u}n^{1/2}+1\Big),$$

$$\int_1^\infty\left(\frac{c}{t}-\frac{d/\sqrt{t}}{n^{1/2}+(d/c)t^{1/2}}\right)dt=2c\ln\Big(\frac{c}{d}n^{1/2}+1\Big)=\frac{2}{11\pi}\ln\Big(\frac{\sqrt{\pi}}{12u}n^{1/2}+1\Big)$$

Simplifying these calculations lead to the result of the theorem.

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